

北一女中 103 學年度《數戰數決》有獎徵答活動



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題號：4-6 頁碼/總頁數：_____ (如果只有一頁，可不填)

$$\begin{aligned} \textcircled{1} \quad & \alpha^5 - \alpha^3 + \alpha = 2 \Rightarrow \alpha(\alpha^4 - \alpha^2 + 1) = 2 \Rightarrow \alpha^4 - \alpha^2 + 1 = \frac{2}{\alpha} \\ \Rightarrow & (\alpha^2 + 1)(\alpha^4 - \alpha^2 + 1) = \frac{2}{\alpha}(\alpha^2 + 1) \Rightarrow \alpha^6 + 1 = 2\left(\alpha + \frac{1}{\alpha}\right) \\ \because & \alpha \in \mathbb{R} \therefore \alpha^6 + 1 > 0 \Rightarrow \alpha > 0 \Rightarrow \alpha + \frac{1}{\alpha} \geq 2, \text{ 但 } \alpha \neq 1 \\ \Rightarrow & \alpha + \frac{1}{\alpha} > 2 \Rightarrow \alpha^6 + 1 = 2\left(\alpha + \frac{1}{\alpha}\right) > 4 \Rightarrow \alpha^6 > 3 \end{aligned}$$

② 考慮 $f(x) = x^5 - x^3 + x$, 則 $f(\alpha) = \alpha^5 - \alpha^3 + \alpha = 2$

$$\text{令 } \beta = 4^{\frac{1}{6}} = \sqrt[3]{2} > 1$$

$$\begin{aligned} f(\beta) - f(\alpha) &= \beta^5 - \beta^3 + \beta - \alpha^5 + \alpha^3 - \alpha \\ &= (\beta - \alpha)(\beta^4 + \beta^3\alpha + \beta^2\alpha^2 + \beta\alpha^3 + \alpha^4 - \beta^2 - \beta\alpha - \alpha^2 + 1) \\ &= (\beta - \alpha) [\beta^2(\beta^2 - 1) + \alpha^2(\alpha^2 - 1) + \beta\alpha(\beta\alpha - 1) + \beta^3\alpha + \beta\alpha^3 + 1] \end{aligned}$$

$\because \alpha^6 > 3, \beta = 4^{\frac{1}{6}} \therefore \alpha, \beta > 1 \Rightarrow$ 上式中括號為正

$\Rightarrow f(\beta) - f(\alpha)$ 和 $\beta - \alpha$ 同號

$$\begin{aligned} \text{又 } f(\beta) - f(\alpha) &= \sqrt[3]{32} - 2 + \sqrt[3]{2} - (\alpha^5 - \alpha^3 + \alpha) = \sqrt[3]{32} + \sqrt[3]{2} - 2 - 2 \\ &> \sqrt[3]{27} + \sqrt[3]{1} - 2 - 2 = 0 \end{aligned}$$

$$\Rightarrow \beta - \alpha > 0 \Rightarrow 4^{\frac{1}{6}} - \alpha > 0 \Rightarrow \alpha^6 < 4$$

由 ①、② 知 $3 < \alpha^6 < 4$